

AS759 - Homework #5
Due: No later than 4:00pm on Thursday, December 19

Note: You are welcome to use tables of integrals and/or symbolic integrators when solving all problems. Just be sure to note down the generic form of the integral if you solve a problem in this way.

1. **(20 points)** The gravitational lens system known as PG1115 is observed to have an Einstein ring of radius 0.6 arcsec. The source is located at a distance of 1750 Mpc from the earth, the lens is located at a distance of 940 Mpc from the earth, and the distance between the lens and the source is 1290 Mpc. Suppose that the 3-dimensional mass distribution of the lens is given by:

$$\rho(r) = \frac{3a^2}{4\pi} \frac{\mathcal{M}}{(r^2 + a^2)^{5/2}}$$

where \mathcal{M} is the total mass of the lens and a is a constant.

a) Compute the surface mass density of the lens, $\Sigma(R)$.

b) Recall that the lens equation is

$$\beta = \theta - \frac{D_{ds}}{D_s} \hat{\alpha}$$

and in the case of a spherically-symmetric lens

$$\hat{\alpha} = \frac{4GM(R)}{c^2 R}.$$

Compute the mass of the lens interior to the Einstein ring, and express your final answer in units of solar masses. What type of astronomical object is the lens?

c) In order for a lens to produce an Einstein ring, its central surface mass density must be at least

$$\Sigma_c \equiv \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}}$$

If the ratio of the central surface mass density of the lens to the above critical surface mass density is $\Sigma(0)/\Sigma_c = f$, what must the value of f be such that the total mass of the lens is finite?

2. (20 points) The convergence of a gravitational lens is given by:

$$\kappa(\vec{\theta}) \equiv \frac{\Sigma(\vec{\theta})}{\Sigma_c}, \quad \Sigma_c \equiv \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}}.$$

In the case of an axisymmetric lens, $\kappa(\vec{\theta}) = \kappa(\theta)$ and it can be shown that the magnitude of the shear is given by:

$$\gamma(\theta) = \frac{\bar{\Sigma}(\theta) - \Sigma(\theta)}{\Sigma_c},$$

where $\bar{\Sigma}(\theta)$ is the mean surface mass density of the lens interior to θ .

- a) Consider a singular isothermal sphere lens, for which the 3-dimensional mass density is:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

(as usual, σ_v is the line of sight velocity dispersion). Compute $\kappa(\theta)$ and $\gamma(\theta)$ for the singular isothermal sphere, and the ratio $\kappa(\theta)/\gamma(\theta)$.

- b) To first order, an intrinsically circular source galaxy will be transformed into an ellipse by gravitational lensing. In particular, a circular source with unit radius will be transformed into an ellipse via a stretching of the image in two directions: (1) along the radius vector that connects the centroids of the lens and the source and (2) tangential to the radius vector that connects the centroids of the lens and the source. To first order, the radial stretch is $(1 - \kappa + \gamma)^{-1}$ and the tangential stretch is $(1 - \kappa - \gamma)^{-1}$. To first order, then, by what factor will the elliptical image be magnified compared to the source? Returning the isothermal sphere lens, how will the magnification and the induced image ellipticity ($\epsilon = 1 - b/a$) vary with distance from the lens center, θ ?

3. (25 points) Consider a “double quasar” that has been lensed by a foreground elliptical galaxy, and two images are observable. The I_+ image is located at $\theta_+ = +1.75''$ and the I_- image is located at $\theta_- = -0.95''$. (The plus sign indicates that the I_+ image and the source are on the same side of the lens, while the minus sign indicates that the I_- image is on the opposite side.) Further suppose that $D_{ds} = 0.5h^{-1}$ Gpc, $D_s = 1.0h^{-1}$ Gpc, the distance between the observer and the lens is $D_d = 0.5h^{-1}$ Gpc, and the redshift of the lens is $z_d = 0.5$. **Throughout the problem approximate the lens as a point mass.**

Rays of light passing by a gravitational lens at a location θ on the sky will take longer to reach the observer than if they were traveling through vacuum. If the lens is both isolated in space and has a spherically-symmetric mass distribution, then for a **given light ray**, the total additional travel time is given by:

$$\tau = \frac{(1 + z_d) D_d D_s}{c D_{ds}} \left[\frac{1}{2}(\theta - \beta)^2 - \psi_{2D}(\theta) \right]$$

where z_d is the redshift of the lens and $\psi_{2D}(\theta)$ is given by:

$$\psi_{2D}(\theta) = \frac{2}{c^2} \frac{D_{ds}}{D_s D_d} \int_{\text{observer}}^{\text{source}} \Phi_{3D}(\theta) dl.$$

That is, $\psi_{2D}(\theta)$ is simply the 3-dimensional gravitational potential of the lens integrated along the line of sight, multiplied by some convenient numbers that make ψ_{2D} a dimensionless quantity. The first term in the expression for τ is the “geometrical” time delay (i.e., the time delay caused solely by the additional path length). The second term is the “gravitational” time delay (i.e., the time delay caused by time dilation in the gravitational field of the lens).

a) Write down expressions for the roots of the quadratic lens equation above (i.e., θ_+ and θ_-) and determine the location of the source, β . Express your final answer in units of arcsec.

b) Compute the value of the Einstein radius, θ_E , in arcsec. How does the value of θ_E compare to the **separation** between the two images?

c) Compute the mass of the lensing galaxy, and express your final answer in units of $h^{-1} M_\odot$. Is this a reasonable mass for an elliptical galaxy?

d) Compute the total additional time compared to vacuum, τ_+ and τ_- , that each of the two light rays must travel in order to reach the observer. Express your final answers in units of h^{-1} years. Compare the values you obtain to the “Hubble age”. Note that for a point mass

$$\psi_{2D}(\theta) = \theta_E^2 \ln(|\theta|).$$

e) What percentage of τ_+ and τ_- in d) is caused by the “geometrical” part of the time delay (i.e., the additional path length caused by the deflection of light)? Note: this is supposed to be a simple estimation.

f) Suppose the quasar were to undergo a large flare. Which of the two images, I_+ or I_- , would show the flare **first**?

g) Suppose you obtain light curves for the two images, and discover that the time delay between one image showing the flare and the other image showing the flare is 125 days. What would you obtain for the value of H_0 ? Express your final answer in units of km/s/Mpc.

4. **(35 points)** In this problem you will explore the shapes of the images that result from lensing by a point mass *pictorially*. Consider a point mass lens with mass $1.0 \times 10^{15} M_{\odot}$. Let the lens have a redshift $z_d = 0.5$ and consider a source galaxy at a redshift $z_s = 1.0$. Take $\Omega_{m0} = 0.25$, $\Omega_{\Lambda} = 0.75$, and $H_0 = 70$ km/s/Mpc, in which case the angular diameter distances are given by $D_d = 1282.6$ Mpc, $D_s = 1705.2$ Mpc, and $D_{ds} = 743.2$ Mpc and the Einstein radius is 52.5 arcsec. Let the source galaxy have an intrinsically circular shape, and an intrinsic radius of 5 arcseconds. We know that every location in the source galaxy will map to TWO locations in the “image plane”; i.e., the two locations θ_{\pm} .

Using a Cartesian coordinate system to represent the observed sky (i.e., ignoring any curvature to the sky), place the lens at the origin of the coordinate system and place the source along the x-axis at the following locations: $x/\theta_E = 0.1, 0.25, 0.5, 0.75, 1.0, 2.0, 5.0$. What you will do below is construct the resulting images for these source locations using a simple prescription.

Take the source galaxy to be a uniform disk, in which case you can use a random number generator to model the source galaxy as a random distribution of points with radius 5 arcseconds, centered on the coordinates above. For each point in the source galaxy, compute where the corresponding two points in the lensed images will appear on the sky. In each case, use enough random points in the source galaxy to represent the resulting images with reasonable resolution. (As the magnification of the images increases you will likely want to use more points.)

For each source location along the x-axis, make a SEPARATE plot of the locations of the lensed images. In each plot, indicate the location of the lens on the sky by the letter L and the location of the source by the letter S. Also, on each plot draw a dotted or dashed line that represents the location of the Einstein ring. Simply place dots on the plot to represent the locations of the points within the source that map onto the image locations. Be sure to use SQUARE axis ratios when you make your plots. In addition to your plots, be sure to hand in SKETCHES of the geometry and a clear statement of the trigonometry that you used to map points in the sources into points in the images. That is, for a point in the source with coordinates (x_s, y_s) , show precisely how it maps to the coordinates (x_+, y_+) and (x_-, y_-) . Do this separately for values of $y_s > 0$ and $y_s < 0$.

Comment upon how the location of the source relative to the lens affects the sizes, shapes and locations of the resulting images.