AS759 - Homework #4

Due: Tuesday, November 26 at start of class

Note: You are welcome to use tables of integrals and/or symbolic integrators when solving all problems. Just be sure to note down the generic form of the integral if you solve a problem in this way.

1. **(15 points)** One of the challenges of modern cosmology is figuring out just how far the dark matter halos of galaxies extend. Since the rotation curves of the disks of spirals are approximately flat out to the very limits of detection, the dark matter must extend beyond the visible disk.

One of the things we know from computer simulations of the formation of structure in the universe is that there is no such thing as a "hard edge" to dark matter halos. Instead, the dark matter in the halos blends smoothly into the rest of the dark matter that pervades the entire universe. In this problem we will make a back-of-the-envelope calculation for the radii of the dark matter halos of galaxies. To do this we'll define the radius of the halo to be the distance from the center of the halo at which the mass density of the halo, $\rho(r)$, is equal to the mean mass density of the universe itself, $\langle\rho\rangle$. That is, the condition $\rho(r_{\rm halo}) = \langle\rho\rangle$ will define the radius of the halo.

In order to make the calculation, we have to choose a value of $\langle\rho\rangle$. To do this, let's assume that we live in a flat, matter-dominated universe. In this case we already know that the mean mass density is given by

$$\langle\rho\rangle = \frac{3H_0^2}{8\pi G}$$

a) Suppose that a typical spiral galaxy has a flat rotation curve with $V_c = 220$ km/s and assume that the mass of the dark matter halo is spherically–symmetric. Compute $M(r)$ for the halo (i.e., the mass contained within radius r) and the corresponding halo mass density, $\rho(r)$.

b) Let $H_0 = 100h$ km/s/Mpc and compute the radius of the halo at which the mass density of the halo equals the mean mass density of the universe. Express your final answer in units of h^{-1} Mpc (i.e., don't assume a particular value for H_0 here).

c) If the value for the radial extent of the halos that you obtained in part b) were typical, what would this have to say about the dark matter halos of M31 and the Milky Way? In other words, is the value that you derived in part b) a "reasonable" estimate of the total extent of the halos of most individual (i.e., non-interacting) galaxies? Notes: you may assume $H_0 = 70$ km/s/Mpc and a distance to M31 of 750 kpc.

2. **(25 points)** In the mid- to late-1990's a trio of astronomers named Julio Navarro, Carlos Frenk and Simon White did a series of computer simulations of the formation of dark matter halos using CDM universes. They discovered a very surprising thing in the course of their work. The mass densities, $\rho(r)$, of CDM halos follow a "universal" density law over a wide range of masses ($10^8 M_\odot$ to $10^{15} M_\odot$). The density law that they derived is known as the Navarro, Frenk and White (or "NFW") density profile and is given by:

$$\frac{\rho(r)}{\rho_c} = \frac{\delta_c}{(r/r_s)(1+r/r_s)^2}$$

where $\rho_c = \frac{3H_0^2}{8\pi G}$ and r_s is a fitting parameter that corresponds to a characteristic length scale within the halo. Like all dark matter halos, CDM halos blend smoothly into the rest of the mass of the universe. For convenience, the "edge" of an NFW halo is defined to be the radius inside which the dark matter mass is virialized, $r_{\rm vir}$. To a good approximation the "virialized mass" of an NFW halo is given by

$$M_{\rm vir} = \left(\frac{4\pi}{3}\right) 200 \rho_c r_{\rm vir}^3$$

We can define a parameter c, known as the concentration parameter, that is the ratio of the virial radius and the characteristic radius, $c \equiv r_{\rm vir}/r_s$, in which case it's straightforward to show

$$\delta_c = \frac{200}{3} \frac{c^3}{[\ln(1+c) - c/(1+c)]}$$

Typical values of c range from 10 to 12 for galaxy-mass halos and 6 to 8 for cluster-mass halos.

a) Using the density law above, compute $M(r)$, the mass contained within a given radius, for the NFW halo model.

b) Using the fact that these are spherically–symmetric halos, show that the NFW halo gives rise to a rotation curve of the following form:

$$\left(\frac{V_c(r)}{V_{\rm vir}}\right)^2 = \frac{1}{x} \frac{\ln(1+cx) - (cx)/(1+cx)}{\ln(1+c) - c/(1+c)}$$

where $x \equiv r/r_{\rm vir}$ and $V_{\rm vir} \equiv V_c(r_{\rm vir}) = \sqrt{GM_{\rm vir}/r_{\rm vir}}$.

c) Observations of gravitational lensing by galaxies suggest that for a typical, bright galaxy $M_{\rm vir} \simeq 1 \times 10^{12} h^{-1} M_\odot$ and $r_{vir} \simeq 180 h^{-1}$ kpc. In the NFW theory, such a halo would have $c \simeq 11$. Make a plot of $V_c(r)$ for such a halo from $r = 0$ to $r = r_{\rm vir}$ assuming $h = 0.7$. Be sure to label the axes of your plot appropriately. Comment upon the curve that you get in comparison to the rotation curves that are typical for bright spiral galaxies (e.g., the Milky Way).

3. **(15 points)** Observationally, the galaxy autocorrelation function (a.k.a. the two-point correlation function) is well-fitted by a power law of the form $\xi(r) = (r/r_0)^{-\gamma}$ over a wide range of physical scales, r. If $r_0 = 5h^{-1}$ Mpc and $\gamma = 1.8$ and if this power law held for all scales (i.e., valid for all r), to what power spectrum of galaxies would this autocorrelation function of galaxies correspond?

Hint: Recall that $P(k)$ and $\xi(r)$ are related through

$$\xi(r) = \left(\frac{1}{2\pi}\right)^3 \int d^3k \; P(k) e^{-i\vec{k}\cdot\vec{r}}$$

so that

$$P(k) = \int d^3r \; \xi(r) \; e^{i\vec{k}\cdot\vec{r}}$$

Potentially Useful Formulas

$$H = \frac{1}{a}\frac{da}{dt} \qquad a = \frac{a_0}{(1+z)} \qquad \Omega = \frac{8\pi G\rho}{3H^2} \qquad q = \frac{1}{H^2}\left(-\frac{1}{a}\frac{d^2a}{dt^2}\right)$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3} \qquad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}$$

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + (1-\Omega_0)(1+z)^2 \qquad \text{(no exotic energy)}$$

$$d_A = \frac{c}{H_0 q_0^2 (1+z)^2}\left[zq_0 + (q_0 - 1)(\sqrt{2q_0 z + 1} - 1)\right]$$

$$H_0 \Delta t = \int_0^z \left(\frac{H_0}{H}\right)\frac{dz'}{1+z'}$$

$$\Theta_g = \int_0^{r_g} \frac{dr}{(1-kr^2)^{1/2}} = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} \qquad \text{(comoving distance to a galaxy)}$$

$$\Theta_H(t) = \int_0^t \frac{c\,dt'}{a(t')} \qquad \Theta_E(t) = \int_t^\infty \frac{c\,dt'}{a(t')}$$

4. **(35 points)** Shown on the last page is an idealized "pie slice" of the universe that runs from $6^h\ 0^m\ 0^s$ of right ascension to $12^h\ 0^m\ 0^s$ of right ascension and $1^\circ\ 0'\ 0''$ of declination to $4^\circ\ 0'\ 0''$ of declination.

a) Estimate the length of the "wall" of galaxies in the pie slice. **(Do not assume a value of Hubble's constant to do this.)**

b) Consider a hypothetical cluster of galaxies for which the observed coordinates and redshifts of 15 of the galaxies in this cluster are listed in the table on the following page. Estimate the redshift of the cluster.

c) To the best of your ability, plot the locations of the cluster galaxies on the pie slice.

d) A simple, back-of-the-envelope calculation of the cluster's mass can be obtained from the virial theorem and will lead to an expression of the form:

$$M = \frac{3\sigma^2 R}{G}$$

where σ is the line-of-sight velocity dispersion of the galaxies and R is the radius of the cluster. Use the table below to estimate the radius of the cluster (based upon its angular extent), and compute its mass using the above equation. **Again, do not assume a particular value of the Hubble constant to do this.** Express your final answer in units of $M_\odot$ and note how your answer depends upon h.

e) From the pie slice, what is the apparent linear extent of the cluster along the line of sight? Compare this to the radius of the cluster that you used in part d) and comment on why plotting the observed recession velocity as a distance coordinate gives a **very** distorted view of the universe.

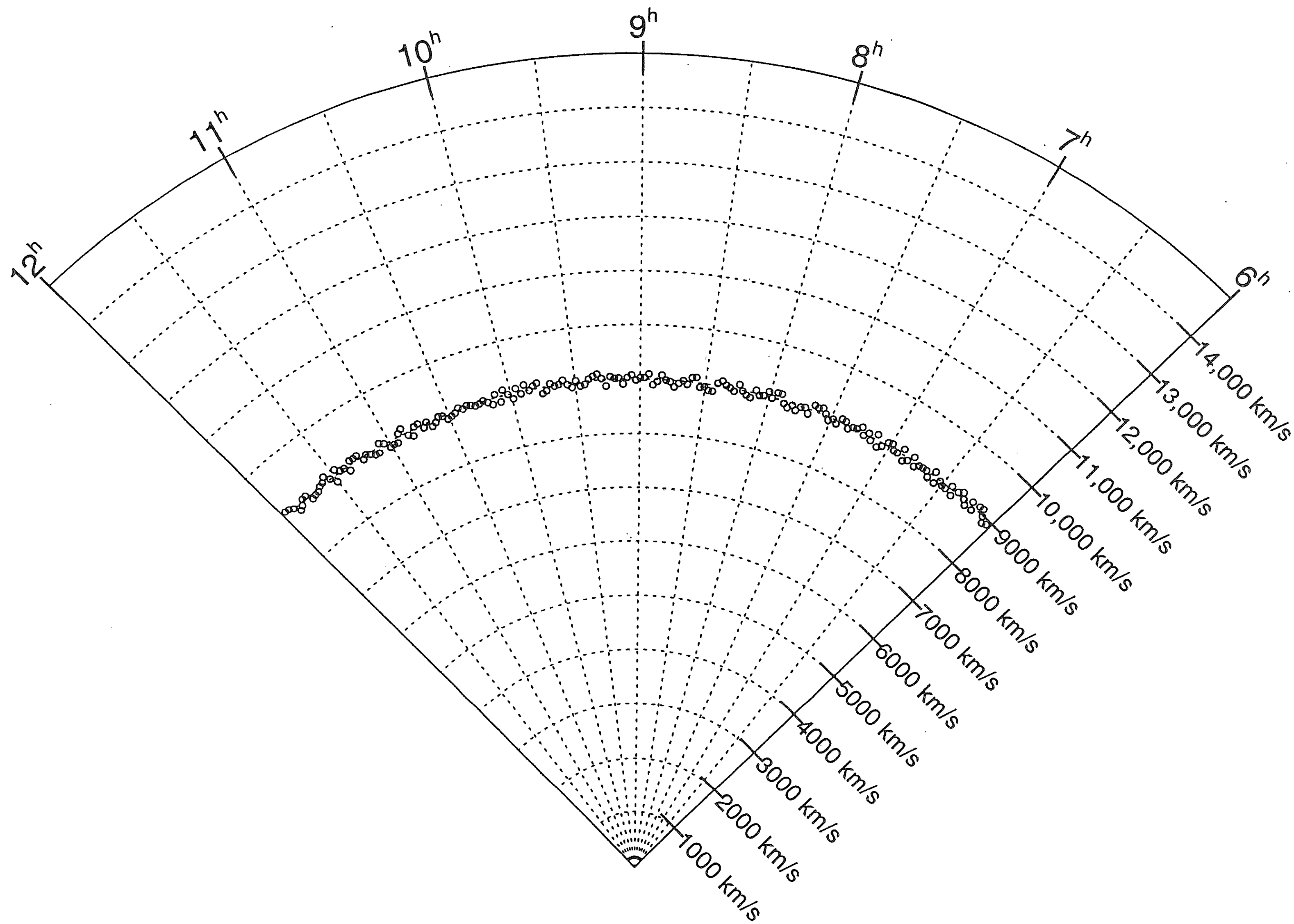
f) The simple, back-of-the-envelope calculation in part d) is not what is used by astronomers to obtain an accurate mass of the cluster. Instead, a careful application of the virial theorem leads to an expression of the form:

$$M = \frac{3\pi\sigma^2 R_e}{2G}, \qquad R_e^{-1} = \frac{1}{N^2}\sum_{i<j}^{N}\frac{1}{|\vec{R}_i - \vec{R}_j|}$$

where $\vec{R}_i$ and $\vec{R}_j$ are coordinate vectors on the sky. As usual, σ is the line-of-sight velocity dispersion. Use this formula to compute the mass of the cluster and compare the mass you obtain here to the mass you computed in part d).

g) In a Gaussian distribution of numbers, you expect that 68% of the numbers will be within 1 standard deviation of the central value (i.e., the mean of the numbers) and 95.4% of the numbers will be within 2 standard deviations of the central value. Based upon this rather simplistic argument, are the radial velocities of the galaxies in the cluster (i.e., in center of mass coordinates) roughly Gaussian–distributed?

RA	DEC	z	RA (degrees)	DEC (degrees)	cz (km/sec)
$8^h\ 1^m\ 7^s$	$3^\circ\ 22'\ 42''$	0.03138	120.28°	3.38°	9414
$7^h\ 59^m\ 42^s$	$2^\circ\ 21'\ 50''$	0.03046	119.93°	2.36°	9138
$8^h\ 3^m\ 19^s$	$2^\circ\ 32'\ 26''$	0.02935	120.83°	2.54°	8805
$8^h\ 2^m\ 34^s$	$3^\circ\ 24'\ 15''$	0.03269	120.64°	3.40°	9807
$8^h\ 1^m\ 10^s$	$3^\circ\ 21'\ 39''$	0.03026	120.29°	3.36°	9078
$8^h\ 0^m\ 43^s$	$3^\circ\ 58'\ 37''$	0.02725	120.18°	3.98°	8175
$7^h\ 58^m\ 39^s$	$3^\circ\ 37'\ 21''$	0.02902	119.66°	3.62°	8706
$8^h\ 0^m\ 33^s$	$3^\circ\ 10'\ 20''$	0.02996	120.13°	3.17°	8988
$7^h\ 57^m\ 15^s$	$3^\circ\ 4'\ 6''$	0.03434	119.31°	3.07°	10302
$7^h\ 57^m\ 1^s$	$3^\circ\ 48'\ 21''$	0.03679	119.25°	3.81°	11037
$7^h\ 59^m\ 15^s$	$3^\circ\ 30'\ 14''$	0.03103	119.81°	3.50°	9309
$8^h\ 2^m\ 11^s$	$2^\circ\ 20'\ 35''$	0.02718	120.55°	2.34°	8154
$7^h\ 59^m\ 53^s$	$3^\circ\ 51'\ 35''$	0.02182	119.97°	3.86°	6546
$7^h\ 59^m\ 39^s$	$2^\circ\ 55'\ 21''$	0.02557	119.91°	2.86°	7671
$7^h\ 58^m\ 24^s$	$2^\circ\ 41'\ 6''$	0.03831	119.60°	2.92°	11493


12h
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9h
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