

AS759 - Homework #3
Due: Tuesday, November 12 at start of class

Note: You are welcome to use tables of integrals and/or symbolic integrators when solving all problems. Just be sure to note down the generic form of the integral if you solve a problem in this way.

1. **(20 points)** The energy per unit volume of a black body radiation field at a temperature T is given by

$$\frac{E}{V} = \frac{4\pi}{c} \int_0^\infty B_\lambda(T) d\lambda$$

Assume that at recombination the universe was in thermal equilibrium and emitted as a blackbody at temperature T , and assume that the number of photons in a comoving volume is conserved. The number of photons with wavelengths between λ and $\lambda + d\lambda$ is given by:

$$N_\gamma = \frac{4\pi}{c} B_\lambda(T) R^3 (hc/\lambda)^{-1} d\lambda$$

where

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}.$$

Show that if the photons that we now observe as the CMBR had a blackbody spectrum at recombination, it will have a black body spectrum today.

2. **(15 points)** Consider a universe in which there is no matter, no radiation, and no cosmological constant. In this case the Friedmann equation can be written as

$$H^2 = -\frac{kc^2}{a^2}$$

and the universe is said to be “curvature driven”. A particularly uninteresting solution to this equation is a flat, static universe. A much more interesting solution is the “Milne universe” for which $k = -1$.

- a) What is $\frac{da}{dt}$ for the Milne universe? Does this universe expand or contract?
- b) What is $a(t)$ for the Milne universe?
- c) What is t_0 , the present age of the universe, for the Milne universe?
- d) What is the proper distance to a galaxy with redshift z in the Milne universe?
- e) Is it possible to have a curvature driven universe with positive curvature? Why or why not?

3. (25 points) Consider the universe in which we appear to live, where $\Omega_{r0} \sim 0$, $\Omega_{m0} = 0.26$ and $\Omega_{\Lambda} = 0.74$.

- a) Compute the age of our universe, t_0 .
- b) Compute the redshift at which the energy density in matter, ρ_m , would have equalled the energy density in the cosmological constant, ρ_{Λ} .
- c) Was the sun in existence when our universe officially became Λ -dominated (i.e., when $\rho_{\Lambda} > \rho_m$)?

4. (20 points) We know that for a matter dominated universe with no cosmological constant the Friedmann equation can be written as:

$$\left(\frac{H}{H_0}\right)^2 = \Omega_0(1+z)^3 + (1-\Omega_0)(1+z)^2.$$

We also know that the mass density of our universe is less than the critical mass density (i.e., $\Omega_0 < 1$). Ignoring the possible presence of a cosmological constant in our universe:

- a) Show that there is a characteristic redshift above which the time evolution of the expansion factor is $a(t) \propto t^{2/3}$. Do not assume a value of Ω_0 here. This is just supposed to be an estimate. Hint: compare $(1+z)$ to $(\Omega_0^{-1} - 1)$.
- b) Similarly, show that there is a characteristic redshift below which the time evolution of the expansion factor is $a(t) \propto t$. Again, do not assume a particular value of Ω_0 here. This, too, is just supposed to be an estimate.
- c) Compute the value of the characteristic redshift for $\Omega_0 = \Omega_{m0} = 0.25$ and, given what we know from problem 3 about when Λ became important in our universe, comment on whether or not it was fair to neglect Λ in parts a) and b). That is, would you expect that to a reasonable approximation the matter-dominated phase of our universe would have shown these two expansion rates?

5. (20 points) A flat, empty universe in which the only contribution to the energy density of the universe is a cosmological constant is known as a de Sitter universe.

- a) What must the value of Ω_{Λ} be for the de Sitter universe?
- b) Compute the value of the deceleration parameter, q , for the de Sitter universe.
- c) Compute the distance to the event horizon in the de Sitter universe.
- d) Compute the proper distance to a galaxy with redshift z in the de Sitter universe.

6. (25 points) The angular diameter distance between two objects at redshifts z_{min} and z_{max} is given by

$$d_A = \frac{2c [1 - \Omega_0 - \sqrt{1 + \Omega_0 z_{min}} \sqrt{1 + \Omega_0 z_{max}}] (\sqrt{1 + \Omega_0 z_{min}} - \sqrt{1 + \Omega_0 z_{max}})}{H_0 \Omega_0^2 (1 + z_{min})(1 + z_{max})^2}$$

in the case that $\Lambda = 0$. For a flat universe in which $\Omega_{m0} + \Omega_\Lambda = 1$, the angular diameter distance between two objects at redshifts z_{min} and z_{max} is given by

$$d_A = \frac{c}{H_0(1 + z_{max})} \int_{1+z_{min}}^{1+z_{max}} \frac{du}{[\Omega_{m0}u^3 + 1 - \Omega_{m0}]^{1/2}}.$$

Consider two galaxies whose redshifts are given by $z_1 = 0.5$ and $z_2 = 1.0$. Consider also three possible universes: [1] an ‘‘Einstein-de Sitter universe’’ in which $\Omega_0 = \Omega_{m0} = 1$ and $\Omega_\Lambda = 0$, [2] an open universe with $\Omega_0 = \Omega_{m0} = 0.25$ and $\Omega_\Lambda = 0$, and [3] a ‘‘concordance’’ universe in which $\Omega_{m0} = 0.25$ and $\Omega_\Lambda = 0.75$.

- a) Compute the angular diameter distances D_1 , D_2 , and $D_{1,2}$ for the Einstein-de Sitter Universe. Here D_1 and D_2 are the angular diameter distances to galaxies 1 and 2 respectively as seen by us (i.e., $z_{min} = 0$). The angular diameter distance $D_{1,2}$ is the angular diameter distance to galaxy 2 as seen by an observer on galaxy 1 (i.e., $z_{min} = 0.5$). How does the value of $D_1 + D_{1,2}$ compare to the value of D_2 ?
- b) Repeat a) for the open universe.
- c) Repeat a) for the concordance universe.
- d) How does D_2 in the open universe compare to D_2 in the Einstein-de Sitter universe? (Compute a ratio.)
- e) Repeat d) for the concordance universe and the Einstein-de Sitter universe.
- f) Comment on the cause of your results in d) and e).

Potentially Useful Formulas

$$H = \frac{1}{a} \frac{da}{dt} \quad a = \frac{a_0}{(1+z)} \quad \Omega = \frac{8\pi G\rho}{3H^2} \quad q = \frac{1}{H^2} \left(-\frac{1}{a} \frac{d^2a}{dt^2} \right)$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda}{3} \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}$$

$$\left(\frac{H}{H_0} \right)^2 = \Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + (1-\Omega_0)(1+z)^2 \quad (\text{no exotic energy})$$

$$d_A = \frac{c}{H_0 q_0^2 (1+z)^2} \left[z q_0 + (q_0 - 1)(\sqrt{2q_0 z + 1} - 1) \right]$$

$$H_0 \Delta t = \int_0^z \left(\frac{H_0}{H} \right) \frac{dz'}{1+z'}$$

$$\Theta_g = \int_0^{r_g} \frac{dr}{(1-kr^2)^{1/2}} = \int_{t_e}^{t_0} \frac{c dt}{a(t)} \quad (\text{comoving distance to a galaxy})$$

$$\Theta_H(t) = \int_0^t \frac{c dt'}{a(t')} \quad \Theta_E(t) = \int_t^\infty \frac{c dt'}{a(t')}$$