The Virial Theorem

Consider a self-gravitating system of N particles. Let $\vec{r}_i$ be the location of particle i (measured from the center of mass). In center of mass coordinates, the particle's velocity will be $\dot{\vec{r}}_i$ and its acceleration will be $\ddot{\vec{r}}_i$.

It can be shown straightforwardly that the equation of motion for particle i is

$$m_i \ddot{\vec{r}}_i = -\frac{1}{2} G m_i \sum_{j \neq i} \frac{m_j (\vec{r}_i - \vec{r}_j)}{d_{ij}^3} \qquad (1)$$

where d_{ij} is the distance between particles i & j. Here m_i is the mass of particle i & m_j is the mass of particle j

The above equation of motion can be obtained by using Lagrange's equations & the Lagrangian

$$\mathcal{L} = \sum_i \frac{1}{2} m_i (\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i) + \frac{1}{2} \sum_{i \neq j} \sum_j \frac{G m_i m_j}{d_{ij}}$$

If we sum both sides of eqn. (1) over all particles we find:

$$\sum_i m_i \ddot{\vec{r}}_i = -\frac{1}{2} G \sum_i m_i \sum_{j \neq i} \frac{m_j (\vec{r}_i - \vec{r}_j)}{d_{ij}^3} \qquad (2)$$

Now take the dot product of 2 with $\vec{r}_i$

If we do this, the left hand side becomes

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \vec{r}_i = \underbrace{\sum_i m_i \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \vec{r}_i\right)}_{\frac{1}{2}\frac{d^2}{dt^2}\sum_i m_i\left(\vec{r}_i \cdot \vec{r}_i\right)} - \underbrace{\sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}_{2T \text{ (twice the KE)}}$$

Since the moment of inertia is $I = \sum_i m_i (\vec{r}_i \cdot \vec{r}_i)$ we have

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \vec{r}_i = \frac{1}{2}\frac{d^2 I}{dt^2} - 2T$$

where T is the KE

The right hand side (after the dot product) becomes

$$-\frac{1}{2} G \sum_i \sum_{j \neq i} \frac{m_i m_j (\vec{r}_i - \vec{r}_j) \cdot \vec{r}_i}{d_{ij}^3}$$

$$= -G \sum_i \sum_{j > i} \frac{m_i m_j \left[(\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)\right]}{d_{ij}^3}$$

if we recall $\vec{r}_i - \vec{r}_j = -(\vec{r}_j - \vec{r}_i)$ & do some manipulations (left to the reader) we have:

$$= -G \sum_i \sum_{j > i} \frac{m_i m_j}{d_{ij}}$$

$= U$, the total potential energy

So by setting LHS = RHS we get

$$\frac{1}{2}\frac{d^2I}{dt^2} - 2T = U$$

or $\frac{1}{2}\frac{d^2I}{dt^2} = 2T + U$, which is known as Lagrange's Identity

in a steady state system, the moment of inertia is a constant (no expansion, no contraction), so $\frac{d^2I}{dt^2} = 0$

$\therefore$ $2T + U = 0$ in a steady state