# Appendix A: Useful Mathematical Techniques

## ***Comparisons: Forming Algebraic Ratios***

It is often the case that you need to compare some characteristic of two objects, for example their brightness, but do not have handy the values of the constants involved or only know a proportionality such as "brightness decreases as one over the square of the distance." In such cases, the easiest method for determining the answer is to form an algebraic ratio from the equation or proportionality. Then, the constants are divided by themselves and therefore cancel out (because a number divided by itself equals one).

*Example 1:*

Star A has a surface temperature (*T*) that is three times hotter than that of star B. The two stars have the same radius (*R*). Compare their luminosities (*L*). [Note: The preferred form of a comparison is to determine how many times greater or smaller the quantity is for one object than for the other.]

Solution: The relevant equation is (6-5):

$$L = 4\pi\sigma T^4 R^2.$$

Here, $\sigma$ is a constant. While you could plug in the value of the constant and try to calculate $L$ for each star, this would fail because you are not given $R$ – you only know that it is the same for each star. The easiest way to solve this problem is to **form an algebraic ratio for both the right- and left-hand sides of the equation:**

$$\frac{L_{\rm A}}{L_{\rm B}} = \frac{4\pi\sigma T_{\rm A}^4 R_{\rm A}^2}{4\pi\sigma T_{\rm B}^4 R_{\rm B}^2}.$$

In other words, divide the entire equation for star A by the equation for star B. The answer, as given on the left-hand side, is the ratio of the luminosity of star A to that of star B, *which is precisely what you need for your final answer to the question!* (This will not always be the case: Sometimes you need to do some algebra to get the ratio of the quantity of interest on the left-hand side.)

Now the task is much easier: You can cancel the $4\pi\sigma$ term since it is divided by itself. Also, since you are given that $R_A = R_B$ in this problem, you can cancel these as well to get

$$\frac{L_{\rm A}}{L_{\rm B}} = \frac{T_{\rm A}^4}{T_{\rm B}^4} = \left(\frac{T_{\rm A}}{T_{\rm B}}\right)^4 = 3^4 = 81,$$

where the numbers correspond to the information given that $T_A$ is 3 times $T_B$. So, the answer, stated in words, is that the luminosity of star A is 81 times that of star B.

*Example 2:*

Star X has the same brightness (*B*) as star Y, but the temperature (*T*) of star X is twice that of star Y. Furthermore, star X is two times more distant (*d*) than star Y. Compare the radius (*R*) of star A with that of star B.

Solution: The relevant equation is (6-7):

$$B = \frac{\sigma T^4 R^2}{d^2}.$$

Since we are interested in comparing radii, we should first solve this equation for $R$, then form an algebraic ratio before plugging in the numbers. As a first step, we multiply both sides of the equation by $[d^2/(\sigma T^4)]$ and switch the left- and right-hand sides:

$$R^2 = \frac{Bd^2}{\sigma T^4}.$$

We now form an algebraic ratio of this equation:

$$\frac{R_X^2}{R_Y^2} = \frac{B_X d_X^2 /(\sigma T_X^4)}{B_Y d_Y^2 /(\sigma T_Y^4)}.$$

Since $\sigma$ is a constant and the brightness is the same for both stars, these two quantities cancel out. Furthermore, for complex divisions as on the right-hand side, the term $(\sigma T_X^4)$ divided into the numerator can become part of the overall denominator, while the term $(\sigma T_Y^4)$ divided into the denominator can become part of the overall numerator. So, we can simplify the equation to the form

$$\frac{R_X^2}{R_Y^2} = \frac{d_X^2 T_Y^4}{d_Y^2 T_X^4}.$$

We need to simplify further by taking the square-root of both sides to get

$$\frac{R_X}{R_Y} = \frac{d_X T_Y^2}{d_Y T_X^2} = \left(\frac{d_X}{d_Y}\right)\left(\frac{T_Y}{T_X}\right)^2 = (2)\left(\frac{1}{2}\right)^2 = \frac{1}{2} = 0.5.$$

So, we find that the radius of star X must be half that of star Y.

*Proportionalities:* A proportionality is a way of expressing the algebraic relation between two or more quantities. It differs from an equation only in that the constants are ignored. Since the constants divide out when forming algebraic ratios, such ratios can be used when given a proportionality. For example, if all you know is that force $F$ is proportional to one divided by the square of the distance (also expressed as "inversely proportional to the distance squared"), and if all other relevant quantities are held constant, you can form the following algebraic ratio:

$$\frac{F_A}{F_B} = \frac{1/d_A^2}{1/d_B^2} = \frac{d_B^2}{d_A^2} = \left(\frac{d_B}{d_A}\right)^2.$$

## ***Dealing with Very Large and Very Small Numbers***

The study of the universe on scales ranging from the entire cosmos to inside the nuclei of atoms by necessity involves both enormous and extremely tiny numbers. Fortunately, there is a very easy way to express these numbers and simple rules for multiplications and divisions involving them.

Scientific Notation

Of course, we could write down huge numbers such as six quadrillion as 6,000,000,000,000,000 and tiny numbers such as 2-trillionths as 0.000000000002, but that is too tedious. Rather, the preferred way of doing it is to use powers of ten to replace the zeroes and decimal places. This is called scientific notation. To convert a large number to scientific notation, simply start with the first digit followed by a decimal point, followed in turn by the remaining digits (we stop when we hit the final set of zeroes). This is then multiplied by 10 raised to the power of the number of digits between the decimal point (either written or implied) of the original number and the 2nd number from the left. This is most easily grasped by looking at some examples:

$6{,}000{,}000{,}000{,}000{,}000 = 6\times10^{15}$
$5{,}280 = 5.28\times10^{3}$
$792{,}320{,}000 = 7.9232\times10^{8}$
$392{,}822.5 = 3.928225\times10^{5}$

For numbers less than one, the rule is to write the significant numbers in the standard form, e.g., 2.345, multiplied times 10 raised to a negative power equal to the number of positions one would need to move the decimal point to get it just to the right of the first significant digit (2 in the above example). Examples of this conversion are:

$0.0002345 = 2.345\times10^{-4}$
$0.000000000917 = 9.17\times10^{-10}$
$0.412 = 4.12\times10^{-1}$

It is sometimes convenient when working on a numerical problem to convert even a number between 1 and 10 to scientific notation. Since $1 = 10^{0}$, this is simple; for example, $6.24 = 6.24\times10^{0}$.

Multiplication and Division

The multiplication of two numbers written in scientific notation follows the simple rules: multiply the two numbers before the "×" sign together and place the result before the "×" sign, then add the powers of ten (i.e., the exponents) of the two numbers. E.g., the product of $2\times10^{3}$ and $4\times10^{17}$ is $8\times10^{20}$ and the product of $3\times10^{6}$ and $3\times10^{-9}$ is $9\times10^{-3}$. The only slightly tricky part is if the product of the two numbers before the "×" sign is greater than 10. In this case, slide the decimal point one position to the left and add one to the power of 10. For example, the product of $4\times10^{8}$ and $5\times10^{-13}$ is $20\times10^{-5} = 2\times10^{-4}$ and the product of $6.50\times10^{14}$ and $2.00\times10^{12}$ is $13.0\times10^{26} = 1.30\times10^{27}$.

Similar rules apply to division, except that the exponent of the divisor (the number on the bottom) is subtracted from the exponent of the dividend (the number on the top). For example:

$$\frac{6.5\times10^{14}}{1.3\times10^{8}} = 5.0\times10^{6}$$

$$\frac{4.4\times10^{6}}{2.0\times10^{-15}} = 2.2\times10^{21}$$

If the resulting number before the "×" sign is less than 1, then move the decimal point one place to the right and subtract 1 from the power of ten of the result. For example:

$$\frac{2.0\times10^{10}}{5.0\times10^{12}} = 0.4\times10^{-2} = 4.0\times10^{-3}$$

***Units***

In general, scientists use standard units: meters (m) for lengths and distances, kilograms (kg) for mass, and seconds (s) for time. This sets the scale for other quantities. For example, velocities are then expressed in m/s and energy is expressed in Joules (J), where 1 J = 1 kg $m^2/s^2$. However, it is sometimes inconvenient to use these units when considering quantities that are either very small or very large. The chapters of this textbook therefore introduce some units that are in common use. In general, you can use these special units in numerical problems as long as you form algebraic ratios as described above. In this case, *you need to be certain to use the same units for similar quantities*. For example, if you use kiloparsecs (kpc) for the distance to one star, you must use kpc for the distance to the other star. If the values given do not have the same units, then one of the values needs to be converted to the units of the other.

*Example:*

Two galaxies are essentially identical, with the same luminosity ($L$). Galaxy A is at a distance ($d$) of 500 kpc, while galaxy B is at a distance of 10 Mpc. Compare the brightness ($B$) of galaxy A with that of galaxy B.

Solution: The relevant equation is the first part of (6-7):

$$B = \frac{L}{4\pi d^2}.$$

As before, we form an algebraic ratio:

$$\frac{B_\mathrm{A}}{B_\mathrm{B}} = \frac{L_\mathrm{A}/(4\pi d_\mathrm{A}^2)}{L_\mathrm{B}/(4\pi d_\mathrm{B}^2)} = \left(\frac{d_\mathrm{B}^2}{d_\mathrm{A}^2}\right) = \left(\frac{d_\mathrm{B}}{d_\mathrm{A}}\right)^2,$$

since we are given that $L_A = L_B$. Before we plug in the numbers, we need to be careful to make sure that the units for distance are the same. As given, they are not, so we need to convert one or the other so that the units are the same. We can choose to express 500 kpc as 0.5 Mpc (which is perhaps more convenient than expressing 10 Mpc as 10,000 kpc, but either conversion is correct). We then have

$$\frac{B_\mathrm{A}}{B_\mathrm{B}} = \left(\frac{d_\mathrm{B}}{d_\mathrm{A}}\right)^2 = \left(\frac{10\ \mathrm{Mpc}}{0.5\ \mathrm{Mpc}}\right)^2 = 20^2 = 400.$$

So, we find that galaxy A is 400 times brighter than galaxy B.